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Generalized Heisenberg algebra: application to the harmonic oscillator

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Abstract

The deformed Poisson algebra recently introduced to investigate integrable systems (2003 *J. Phys. A: Math. Gen.* **36** 12181–203, 2005 *J. Math. Phys.* **46** 042702) is used to perform the transition from the phase space of classical observables (functions depending on positions and momentums) to the Hilbert space of physically well-defined Hermitian operators. A Hamiltonian operator for the harmonic oscillator system is constructed and the eigenvalue problem is solved. The generalization to an n-dimensional space shows that such an algebra does not break the rotational symmetry.

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1. Introduction

Generalizations of the HA, also called deformed Heisenberg algebras, continue to be under a thorough and versatile scrutiny in mathematics and physics [1–11] (and references therein). The concept was first developed by Snyder [2] about 20 years after Heisenberg and Schrödinger invented quantum mechanics around 1925 and Heisenberg discovered the uncertainty relation in 1927. The problem was considered as small additions to the canonical commutation relations. In [2], such an issue was raised in connection with the idea of quantization of the phase space. As mentioned above, all these exciting activities are motivated by fundamental as well as practical considerations. Accordingly, a number of suggestions have been made since the earlier work of Snyder [2]. As a matter of citation, let us just mention q- or (p, q)-deformed HA and their variants, which are relevant in q- or (p, q)-deformed phase spaces. See [8, 12–15] for more details.

The question of deformation of the HA is also approached along purely practical lines when solving eigenvalue problems [1]. For instance, when we have a Hamiltonian in the Schrödinger equation with a potential which does not allow exact analytical solutions to be obtained, one usually reduces it to a familiar form using generalized position and momentum operators which fail to satisfy the HA. The permutation relations between these operators are the so-called deformed relations. By this procedure, we transfer the 'pathological' form of the Hamiltonian into a deformation of HA.

The generalized HA is also induced from a classical mechanics generalization. In this direction, one of the most successful results on the extension of symplectic mechanics is certainly the Nambu proposal [9] as a generalization of Hamiltonian mechanics by considering brackets involving $n \geqslant 3$ functions. The quantization of these brackets has been discussed in the framework of deformation quantization [16]. However, despite the elegance and beauty of Nambu mechanics, it turns out to be somewhat restrictive with many basic problems waiting to be solved when the quantization procedure is applied [10, 11].

Instead of considering a multilinear object, namely the Nambu bracket, which yields a generalization of the Hamiltonian, the aim of this work is to provide a straightforward generalization of HA induced by a bilinear generalized Poisson bracket which has received considerable attention in deformation theories in the last years [17, 18] (and references therein). Such a deformed Poisson bracket has been successfully used to investigate dynamical systems with concrete applications to the Boussineq system as well as to Kaup–Bœ r and Toda systems. Three field and dispersionless systems as well as field soliton and lattice soliton systems have been also examined. The investigations performed by these authors [17] reveal that the number of constructed dispersionless systems is much greater than the number of known soliton systems (dispersive integrable systems). The authors answered to the question whether for any dispersionless Lax hierarchy one can construct a related soliton hierarchy, via a procedure of Weyl–Moyal-like deformation quantization for Poisson algebras of dispersionless systems and appropriate R-matrix theory. They succeeded in finding a unified procedure for the construction of field and lattice Hamiltonian soliton systems in one scheme. Notwithstanding these results relevant for the analysis of integrable systems and the great interest triggered by this bracket in physical applications, to our best knowledge of the literature, a systematic study of its quantization in view of performing the transition from the phase space of classical observables (functions depending on positions and momentums) to the Hilbert space of quantum observables (operators) is still lacking. This paper, in the first part, also fills in this gap, performing a thorough study of associated position and momentum operators, as well as their functional analytic properties, which remain essential to giving as complete and rigourous description as possible of the quantum configuration and momentum spaces. It is worth mentioning that these latter operators yield a generalization of quantum observables such as Hamiltonians of physical systems. The ultimate goal of the second part is to present an exhaustive illustration of such well-defined tools in the interesting case of the harmonic oscillator, which is central to the construction of a number of models in various domains of physics (atomic and condensed matter physics, optics, etc).

The paper is organized as follows. In section 2, we introduce the generalized HA and provide the corresponding representation theory. In section 3, we give an application to the harmonic oscillator. The eigenvalues and eigenfunctions of the deformed Hamiltonian are carried out. Section 4 is devoted to a generalization to n-dimensional spaces. Finally, the paper ends with some concluding remarks in section 5.

2. Generalized Heisenberg algebra

For the convenience of the development, we first briefly recall the classical definition of a two-Poisson manifold, which is a two-dimensional Euclidian space $\mathbb{R}^2$ generated by the position and momentum variables $q \equiv x^1$ and $p \equiv x^2$ and equipped with a Poisson bracket (PB).

By introducing $f(q, p)$ and $g(q, p)$, two arbitrary smooth functions, the PB is defined as

$$\{f, g\} = \left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p}\right). \tag{1}$$

Equation (1) can be expressed in the compact form

$$\{f, g\} = \partial_i f J^{ij} \partial_j g, \tag{2}$$

where J^{ij} are the entries of the unitary simplectic matrix J given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3}$$

Equation (2) does not change under the action of a symplectic transformation $Sp(1)$ on the phase space. As is well known, (2) can also be expressed as

$$\{f, g\} = \{x^i, x^j\}\partial_i f \partial_j g, \tag{4}$$

so that, if we know the PB between the generators x^i, we can compute the PB between any pair of functions f and g.

Let us rapidly recall here that, introducing a q_0-PB and requiring that it must be invariant under the action of the q_0-symplectic group $Sp_{q_0}(1)$, we are lead to the following q_0-generators $\hat{x}^i$ [19]:

$$\{\hat{x}^i, \hat{x}^j\}_{q_0} = \hat{\partial}_q \hat{x}^i \hat{\partial}_p \hat{x}^j - q_0^2 \hat{\partial}_p \hat{x}^i \hat{\partial}_q \hat{x}^j. \tag{5}$$

It is easy to verify the following fundamental relations:

$$\{\hat{q}, \hat{q}\}_{q_0} = \{\hat{p}, \hat{p}\}_{q_0} = 0 \tag{6}$$

$$\{\hat{q}, \hat{p}\}_{q_0} = 1 \tag{7}$$

$$\{\hat{p}, \hat{q}\}_{q_0} = -q_0^2, \tag{8}$$

which coincide with the one obtained in [23]. In particular, from (7) and (8) it follows that the q_0-PB is not antisymmetric. A similar behaviour also appears in quantum q_0-oscillator theory [21, 22].

In the same vein and to preserve some consistency in the deformation of the symplectic structure of the phase space, let us now consider a classical mechanical system described by a generalized Poisson bracket involving a parameter r (instead of q_0) and defined by [17]

$$\{f, g\}_{\mathrm{PB}}^r = p^r \left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p}\right), \tag{9}$$

where $r \in \mathbb{N}$. Indeed, one can readily check that (9) is bilinear, antisymmetric and verifies the Leibnitz identity as well as the Jacobi identity. The latter identity can be proved as follows. Let f, g and h be three arbitrary smooth functions. Then, the relation

$$\begin{aligned}\left\{\{f, g\}_{\mathrm{PB}}^r, h\right\}_{\mathrm{PB}}^r &= p^{2r}\left(\frac{\partial^2 f}{\partial q^2}\frac{\partial g}{\partial p}\frac{\partial h}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial^2 g}{\partial q^2}\frac{\partial h}{\partial p}\right) + p^{2r}\left(\frac{\partial f}{\partial q}\frac{\partial^2 g}{\partial q \partial p}\frac{\partial h}{\partial p} - \frac{\partial^2 f}{\partial q \partial p}\frac{\partial g}{\partial q}\frac{\partial h}{\partial p}\right) \\ &\quad - r p^{2r-1}\left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}\frac{\partial h}{\partial q} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\frac{\partial h}{\partial q}\right) - p^{2r}\left(\frac{\partial^2 f}{\partial q \partial p}\frac{\partial g}{\partial p}\frac{\partial h}{\partial q} - \frac{\partial f}{\partial p}\frac{\partial^2 g}{\partial q \partial p}\frac{\partial h}{\partial q}\right) \\ &\quad - p^{2r}\left(\frac{\partial f}{\partial q}\frac{\partial^2 g}{\partial p^2}\frac{\partial h}{\partial q} - \frac{\partial^2 f}{\partial p^2}\frac{\partial g}{\partial q}\frac{\partial h}{\partial q}\right)\end{aligned} \tag{10}$$

can be cyclically manipulated to yield

$$\left\{\{f, g\}_{\mathrm{PB}}^r, h\right\}_{\mathrm{PB}}^r + \left\{\{g, h\}_{\mathrm{PB}}^r, f\right\}_{\mathrm{PB}}^r + \left\{\{h, f\}_{\mathrm{PB}}^r, g\right\}_{\mathrm{PB}}^r = 0. \tag{11}$$

From (9), we obtain

$$\{q, p\}^r_{\mathrm{PB}} = p^r. \tag{12}$$

The case $r = 0$ yields the standard Poisson bracket (1).

The deformed Poisson bracket (9) has become the focus of increasing interest. Indeed, in [17], Blaszak *et al* applied a Weyl–Moyal-like deformation to a systematic construction of the field and lattice integrable soliton systems from Poisson algebras of dispersionless systems, endowed with (9). More recently [18], the same bracket has been used by Szablikowski *et al* to study deformations of standard R-matrices for integrable infinite-dimensional systems.

In the following, we aim at building operators Q and P, quantum counterparts of q and p, respectively, such that a quantum algebra corresponding to (12) and generalizing the usual Heisenberg algebra can be written as

$$[Q, P] = \mathrm{i}\nu_r P^r, \tag{13}$$

where ν_r is a formal parameter with dimension $[\nu_r] = [\hbar]^{1-r}[L]^r$, where $[L]$ is the unit of length and $\hbar = h/(2\pi)$, h being the Planck constant.

The main difference between using (12) instead of (5) and its other variants comes from the fact that the latter, involving the parameter q_0, is no longer a Poisson bracket, and therefore the following quantization rule (which we suppose at least to be satisfied),

$$\frac{1}{i\alpha}[Q_f, Q_g] = Q_{\{f,g\}_{q_0}} \tag{14}$$

(α being a parameter and Q_f, the quantum counterpart of f), cannot be applied to it. Contrarily, one can readily check that (13) used in this work well satisfies (14) for the generators, replacing the right-hand bracket by (9) and α by ν_r. Besides, as a general feature, the parameter q_0, in all the above-mentioned q_0-deformed models, acts as a multiplicative factor in the classical (see for instance (5)) as well as in the quantum description, while in the r deformation given in (12), the parameter r behaves as a simple power in classical mechanics, and as the order of differential operators in the quantum configuration space.

2.1. Position and momentum operators

To satisfy the criteria of the existence of a complete system of (generalized) eigenfunctions, which is fundamental for the physical interpretation of observables, one requires the operators Q and P to be Hermitian [20] (and references therein).

We first consider the configuration space $\{|q\rangle\}$ and the space of smooth complex valued functions with compact support, $\mathcal{C}_0^\infty(\Omega)$, of variable q, Ω being an open set of $\mathbb{R}$. Let Υ be the Hilbert space $L^2(\Omega, \mathrm{d}q)$, equipped with the scalar product

$$\langle u, v\rangle = \int_\Omega u^*(q)v(q)\,\mathrm{d}q \qquad \forall u, v \in \Upsilon, \tag{15}$$

where u^* is the complex conjugate of u. $\mathcal{C}_0^\infty(\Omega)$ is dense in Υ. One has the following.

Theorem 1. *Let Q and P be the operators defined as*

$$\begin{aligned} &Q, P : \mathcal{C}_0^\infty(\Omega) \subset L^2(\Omega, \mathrm{d}q) \longrightarrow L^2(\Omega, \mathrm{d}q) \\ &Q\phi(q) = (-\mathrm{i})^r \hbar^{r-1}\nu_r \left(q\frac{\mathrm{d}^r\phi(q)}{\mathrm{d}q^r} + \frac{r}{2}\frac{\mathrm{d}^{r-1}\phi(q)}{\mathrm{d}q^{r-1}}\right) \\ &P\phi(q) = -\mathrm{i}\hbar\frac{\mathrm{d}\phi(q)}{\mathrm{d}q} \end{aligned} \tag{16}$$

$\forall\phi \in \mathcal{C}_0^\infty(\Omega)$. *Then, Q and P are Hermitian and verify algebra (13).*

Proof. Let $\phi, \psi \in \mathcal{C}_0^\infty(\Omega)$. From the definition of the scalar product (15), we get

$$\begin{aligned}\langle Q\phi, \psi\rangle &= \mathrm{i}^r\hbar^{r-1}\nu_r \int_\Omega \left(q\frac{\mathrm{d}^r\phi^*(q)}{\mathrm{d}q^r} + \frac{r}{2}\frac{\mathrm{d}^{r-1}\phi^*(q)}{\mathrm{d}q^{r-1}}\right)\psi(q)\,\mathrm{d}q \\ &= \int_\Omega \phi^*(q)\left[(-\mathrm{i})^r\hbar^{r-1}\nu_r\left(q\frac{\mathrm{d}^r\psi(q)}{\mathrm{d}q^r} + \frac{r}{2}\frac{\mathrm{d}^{r-1}\psi(q)}{\mathrm{d}q^{r-1}}\right)\right]\mathrm{d}q \\ &= \langle\phi, Q\psi\rangle, \end{aligned} \tag{17}$$

where use has been made of integration by parts in the sense of distributions. We readily obtain $Q^\dagger = Q$ while $P^\dagger = P$ is obvious. □

In the same vein, let us now consider the Hilbert space $\Xi = L^2(\Omega, \mathrm{d}p)$ in the momentum space $\{|p\rangle\}$. We have the following.

Theorem 2. *Let Q and P be the operators defined as*

$$\begin{aligned} &Q, P : \mathcal{C}_0^\infty(\Omega) \subset L^2(\Omega, \mathrm{d}p) \longrightarrow L^2(\Omega, \mathrm{d}p) \\ &Q\phi(p) = \mathrm{i}\nu_r\left(p^r\frac{\mathrm{d}\phi(p)}{\mathrm{d}p} + \frac{r}{2}p^{r-1}\phi(p)\right) \\ &P\phi(p) = p\phi(p) \end{aligned} \tag{18}$$

$\forall\phi \in \mathcal{C}_0^\infty(\Omega)$. *Then, Q and P are Hermitian and satisfy (13).*

Proof. It proceeds in the same manner as for theorem 1. □

The case $r = 0$ in prescriptions (16), (18) and (13) yields the usual canonical commutation relation as it should:

$$[Q, P] = \mathrm{i}\hbar I. \tag{19}$$

Postulates (13) and (19) reflect the fact that for macroscopic systems, the pairs P and Q go over into ordinary dynamical variables p and q, and hence commute; their noncommutative operator behaviour must be taken into account at the atomic level, where $\hbar$ becomes significant. Remark that $\mathrm{i}\hbar I$ is a scalar, and therefore invariant under a unitary transformation (since it will commute with any operator), while $\mathrm{i}\nu_r P^r$ is a r-order differential operator which does not commute with non-trivial operators. This feature further highlights the noncommutative character of the geometry described by (13), as we will see in the following.

2.2. *Functional analysis of the position operator*

Here, we deal with the functional analysis of the operator Q.

2.2.1. Configuration space. In the configuration space, the domain of the adjoint, $Q^\dagger$, of the position operator Q is formally defined as [24]

$$D(Q^\dagger) = \left\{ v \in \Upsilon; \exists\psi \in \Upsilon / \langle v, Qf\rangle = \langle\psi, f\rangle \forall f \in \mathcal{C}_0^\infty(\Omega)\right\}. \tag{20}$$

In order to study the self-adjoint extension of Q, we have to refine the definition of $D(Q^\dagger)$. Let $v \in D(Q^\dagger)$. By definition (20), there exists $\psi \in \Upsilon$ such that $\langle v, Qf\rangle = \langle\psi, f\rangle$ for all $f \in \mathcal{C}_0^\infty(\Omega)$. Explicit computation gives

$$\begin{aligned}\langle v, Qf\rangle &= (-\mathrm{i})^r\hbar^{r-1}\nu_r\int_\Omega v^*(q)\left(q\frac{\mathrm{d}^r f(q)}{\mathrm{d}q^r} + \frac{r}{2}\frac{\mathrm{d}^{r-1}f(q)}{\mathrm{d}q^{r-1}}\right)\mathrm{d}q \\ &= (-\mathrm{i})^r\hbar^{r-1}\nu_r\int_\Omega\left((-1)^r\frac{\mathrm{d}^r(qv^*(q))}{\mathrm{d}q^r} + \frac{r}{2}(-1)^{r-1}\frac{\mathrm{d}^{r-1}(v^*(q))}{\mathrm{d}q^{r-1}}\right)f(q)\,\mathrm{d}q \\ &= \langle Qv, f\rangle, \end{aligned} \tag{21}$$

where we have used integration by parts and the conditions

$$\frac{\mathrm{d}^\alpha v(q)}{\mathrm{d}q^\alpha}, \frac{\mathrm{d}^\beta (qv(q))}{\mathrm{d}q^\beta} \in \Upsilon, \tag{22}$$

for $|\alpha| \leqslant r-2$ and $|\beta| \leqslant r-1$. Thus, $\langle v, Qf\rangle = \langle Qv, f\rangle = \langle \psi, f\rangle, \forall f \in \mathcal{C}_0^\infty(\Omega)$, which implies $Qv \in \Upsilon$. It follows that

$$D(Q^\dagger) = \left\{ v \in L^2(\Omega, \mathrm{d}q); \frac{\mathrm{d}^\alpha v}{\mathrm{d}q^\alpha}, \frac{\mathrm{d}^\beta (qv)}{\mathrm{d}q^\beta}, \left(q\frac{\mathrm{d}^r}{\mathrm{d}q^r} + \frac{r}{2}\frac{\mathrm{d}^{r-1}}{\mathrm{d}q^{r-1}}\right) v \in L^2(\Omega, \mathrm{d}q); \right.$$
$$\left. |\alpha| \leqslant r-2, |\beta| \leqslant r-1 \right\}, \tag{23}$$

which is the rigourous mathematical definition of the domain of $Q^\dagger$ in the configuration space.

The eigenvalue problem for the position operator takes, in the configuration space, the form of the r-order differential equation

$$(-\mathrm{i})^r \hbar^{r-1} \nu_r \left(q\frac{\mathrm{d}^r}{\mathrm{d}q^r} + \frac{r}{2}\frac{\mathrm{d}^{r-1}}{\mathrm{d}q^{r-1}}\right)\phi_\lambda(q) = \lambda\phi_\lambda(q), \tag{24}$$

which can be explicitly solved for particular values of r. For $r = 1$, we obtain the eigenfunctions

$$\phi_\lambda(q) = kq^{(-\frac{1}{2}+\frac{\mathrm{i}\lambda}{\nu_1})}. \tag{25}$$

Taking $\Omega = \mathbb{R}, \phi_{\lambda=i}$ and $\phi_{\lambda=-i}$ belong neither to $L^2(\Omega, \mathrm{d}q)$ nor, in particular, to $D(Q^\dagger)$. Therefore, the operator $Q^\dagger$ has empty deficiency subspaces [25]:

$$n_\pm(Q) := \ker(Q^\dagger \pm \mathrm{i}I).D(Q^\dagger). \tag{26}$$

As the deficiency indices in this case are (0, 0), we conclude that the position operator Q is essentially self-adjoint.

2.2.2. Momentum space. In the momentum space, the domain of $Q^\dagger$ is formally defined as

$$D(Q^\dagger) = \left\{v \in \Xi; \exists\psi \in \Xi/\langle v, Qf\rangle = \langle\psi, f\rangle \forall f \in \mathcal{C}_0^\infty(\Omega)\right\}. \tag{27}$$

In a suitable way, let $v \in D(Q^\dagger)$. By definition, there exists $\psi \in \Xi$ such that $\langle v, Qf\rangle = \langle\psi, f\rangle$ for all $f \in \mathcal{C}_0^\infty(\Omega)$. Using integration by parts and the condition $p^r v \in \Xi$, we deduce

$$\begin{aligned}\langle v, Qf\rangle &= \int_\Omega v^*(p)\left[\mathrm{i}\nu_r\left(p^r\frac{\mathrm{d}f(p)}{\mathrm{d}p} + \frac{r}{2}p^{r-1}f(p)\right)\right]\mathrm{d}p \\ &= \mathrm{i}\nu_r\int_\Omega\left(-\frac{\mathrm{d}(p^r v^*(p))}{\mathrm{d}p} + \frac{r}{2}p^{r-1}v^*(p)\right)f(p)\,\mathrm{d}p \\ &= \langle Qv, f\rangle.\end{aligned} \tag{28}$$

Thus, $\langle v, Qf\rangle = \langle Qv, f\rangle = \langle\psi, f\rangle\forall f \in \mathcal{C}_0^\infty(\Omega)$ which implies that $Qv \in \Xi$. Hence,

$$D(Q^\dagger) = \left\{v \in L^2(\Omega, \mathrm{d}p); p^r v, \left(p^r\frac{\mathrm{d}}{\mathrm{d}p} + \frac{r}{2}p^{r-1}\right)v \in L^2(\Omega, \mathrm{d}p)\right\}. \tag{29}$$

The eigenvalue problem for the position operator, in the momentum space, is given by a first-order differential equation

$$\mathrm{i}\nu_r\left(p^r\frac{\mathrm{d}}{\mathrm{d}p} + \frac{r}{2}p^{r-1}\right)\psi_\lambda(p) = \lambda\psi_\lambda(p), \tag{30}$$

which can be immediately solved to obtain eigenfunctions

$$\psi_\lambda(p) = \begin{cases} \dfrac{c}{p^{r/2}} \exp\left(-\dfrac{\mathrm{i}\lambda}{\nu_r(1-r)}\dfrac{1}{p^{r-1}}\right) & \text{if} \quad r \neq 1 \\ c p^{-(\frac{1}{2}-\frac{\mathrm{i}\lambda}{\nu_1})} & \text{if} \quad r = 1. \end{cases} \tag{31}$$

We note that if $\Omega =]0, +\infty[$, there is no eigenvector associated with each of the values $\lambda = \pm\mathrm{i}$. Hence, the operator Q is essentially self-adjoint.

Let us note that, with $r = 1$, the position operator Q (16) defined in the configuration space is exactly the same operator as Q in the momentum space (18), except for a sign and the name of the argument of the wavefunction. Remark also that the choice $\Omega = \mathbb{R}$ in the subsubsection 2.2.1 and $\Omega = \mathbb{R}^+\backslash\{0\}$ in the subsubsection 2.2.2 is not intrinsic to the position operator Q, but because P should be simultaneously essentially self-adjoint. For instance, in the configuration space, the indicial equation for P leads to the solutions $\phi_{\pm i}(q) = k\exp(\mp q/\hbar)$ which do not belong to $L^2(\mathbb{R}, \mathrm{d}q)$, proving that P is also essentially self-adjoint, as it should.

From now on, we assume $\Omega =]0, +\infty[$.

2.3. Relevant properties

As a straightforward application of (13), we have the following properties.

Proposition 1. $\forall n \geqslant 1, \forall t \in \mathbb{R}$:

$$\text{(i)} \quad [Q^n, P] = \mathrm{i}\nu_r \sum_{j=0}^{n-1} Q^j P^r Q^{n-j-1}$$

$$\text{(ii)} \quad [Q, P^n] = \mathrm{i}n\nu_r P^{r+n-1}$$

$$\text{(iii)} \quad [Q, \exp(\mathrm{i}tP)] = -\nu_r t P^r \exp(\mathrm{i}tP)$$

$$\text{(iv)} \quad [Q, P^{-1}] = -\mathrm{i}\nu_r P^{r-2}$$

$$\text{(v)} \quad [\exp(\mathrm{i}tQ), P] = \mathrm{i}\nu_r \sum_{j=1}^{+\infty}\sum_{k=0}^{j-1} \frac{(\mathrm{i}t)^j}{j!} Q^k P^r Q^{j-k-1}$$

$$\text{(vi)} \quad [Q^{-1}, P] = -\mathrm{i}\nu_r Q^{-1} P^r Q^{-1}.$$

Theorem 3 (Heisenberg uncertainty relation)**.** *Let ΔP, ΔQ be the uncertainties in Q and P, respectively, $\langle A\rangle_\psi$ be the mean value of an operator A in the state $|\psi\rangle$. Then*

$$\Delta P \Delta Q \geqslant \nu_r |\langle P^r\rangle_\psi|/2. \tag{32}$$

Proof. Let us set $|\phi\rangle = (Q' + \mathrm{i}\lambda_r P')|\psi\rangle$, where

$$\lambda_r = \frac{\langle P^r\rangle_\psi}{2(\Delta P)^2} \qquad P' = P - \langle P\rangle_\psi \qquad Q' = Q - \langle Q\rangle_\psi. \tag{33}$$

Then we get

$$\begin{aligned} \langle\phi|\phi\rangle &= \langle\psi|(Q' - \mathrm{i}\lambda_r P')(Q' + \mathrm{i}\lambda_r P')|\psi\rangle \\ &= \langle\psi|(Q')^2|\psi\rangle + \mathrm{i}\lambda_r\langle\psi|[Q', P']|\psi\rangle + \lambda_r^2\langle\psi|(P')^2|\psi\rangle, \end{aligned} \tag{34}$$

or equivalently

$$\langle\phi|\phi\rangle = \langle(Q')^2\rangle_\psi + \mathrm{i}\lambda_r\langle[Q', P']\rangle_\psi + \lambda_r^2\langle(P')^2\rangle_\psi.$$

Since $[Q', P'] = [Q, P]$, one deduces

$$\langle\phi|\phi\rangle = \langle (Q')^2\rangle_\psi - \lambda_r \nu_r \langle P^r\rangle_\psi + \lambda_r^2 \langle (P')^2\rangle_\psi. \tag{35}$$

Hence the rhs expression is positive, which implies (32). □

The normalized eigenstates $|\psi\rangle$ ($\langle\psi|\psi\rangle = 1$) for the operators $Q + \mathrm{i}\lambda_r P$ are minimum uncertainty states for the operators Q and P. One can readily check that they verify the relation

$$(Q' + \mathrm{i}\lambda_r P')|\psi\rangle = 0. \tag{36}$$

In momentum representation, this takes the form of the differential equation

$$\left[\mathrm{i}\nu_r\left(p^r \frac{\mathrm{d}}{\mathrm{d}p} + \frac{r}{2} p^{r-1}\right) - \langle Q\rangle + \mathrm{i}\lambda_r (p - \langle P\rangle)\right]|\psi\rangle = 0, \tag{37}$$

which can be solved to obtain

(i) if $r \notin \{1, 2\}$

$$\psi(p) = c_r p^{-r/2} \exp\left[\frac{\lambda_r}{\nu_r (r-2)} p^{-r+2} - \frac{1}{r-1}\left(\frac{\lambda_r}{\nu_r}\langle P\rangle + \frac{1}{\mathrm{i}\nu_r}\langle Q\rangle\right) p^{-r+1}\right], \tag{38}$$

with

$$\begin{aligned} c_r &= \left[\frac{1}{r-1}\sum_{k=0}^{+\infty}\frac{(\alpha_r)^k}{k!}\left(\frac{1}{\beta_r}\right)^{\frac{k(r-2)}{r-1}+1}\Gamma\left(\frac{k(r-2)}{r-1}+1\right)\right]^{-1/2} \\ \alpha_r &= \frac{2\lambda_r}{\nu_r (r-2)} \\ \beta_r &= \frac{2\lambda_r \langle P\rangle}{\nu_r (r-1)}. \end{aligned} \tag{39}$$

(ii) if $r = 1$

$$\psi(p) = c_1 p^{-\frac{1}{2} + (\frac{\lambda_1}{\nu_1}\langle P\rangle + \frac{1}{\mathrm{i}\nu_1}\langle Q\rangle)} \exp\left(-\frac{\lambda_1}{\nu_1} p\right), \tag{40}$$

with

$$c_1 = \left[\left(\frac{\nu_1}{2\lambda_1}\right)^{\frac{2\lambda_1\langle P\rangle}{\nu_1}} \Gamma\left(\frac{2\lambda_1\langle P\rangle}{\nu_1}\right)\right]^{-1/2}. \tag{41}$$

(iii) if $r = 2$

$$\psi(p) = c_2 p^{-(1+\frac{\lambda_2}{\nu_2})} \exp\left[-\left(\frac{\lambda_2}{\nu_2}\langle P\rangle + \frac{1}{\mathrm{i}\nu_2}\langle Q\rangle\right) p^{-1}\right], \tag{42}$$

with

$$c_2 = \left[\left(\frac{\nu_2}{2\lambda_2\langle P\rangle}\right)^{\frac{2\lambda_2}{\nu_2}+1} \Gamma\left(\frac{2\lambda_2}{\nu_2} + 1\right)\right]^{-1/2}. \tag{43}$$

It is noteworthy to point out some remarkable features of theses states. As a matter of fact, consider for instance $r = 2$. Then the uncertainty relation (32) implies $\Delta Q \geqslant \nu_2 \Delta P/2$; when ΔP is large, ΔQ is also large.

3. Application to the harmonic oscillator

In this section, we deal with the application of the generalized HA (13) to the harmonic oscillator. Ehrenfest's theorem is stated. Then, we give an explicit expression of the one-dimensional Hamiltonian of the harmonic oscillator in momentum space and solve the corresponding eigenvalue problem.

The generalized Hamiltonian of the quantum harmonic oscillator reads

$$\mathcal{H} = \frac{P^2}{2m} + \frac{m\omega^2}{2}Q^2, \tag{44}$$

which can be used together with (13) to state the following.

Theorem 4 (Ehrenfest's theorem)**.** *Given algebra (13) and the Hamiltonian (44), the mean value temporal evolution of Q and P satisfies the equations*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle Q\rangle(t) &= \frac{\nu_r}{\hbar m}\langle P^{r+1}\rangle \\ \frac{\mathrm{d}}{\mathrm{d}t}\langle P\rangle(t) &= -\frac{\omega^2\nu_r m}{2\hbar}\langle P^r Q + QP^r\rangle.\end{aligned} \tag{45}$$

Proof. Given an observable A, the time evolution of its mean value with respect to the state $|\psi(t)\rangle$ becomes [26]

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle(t) = \frac{1}{\mathrm{i}\hbar}\langle[A,\mathcal{H}]\rangle + \left\langle\frac{\partial A}{\partial t}\right\rangle. \tag{46}$$

Applying this definition and taking into account the fact that the operators Q and P do not explicitly depend on t, one immediately gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Q\rangle(t) = \frac{1}{\mathrm{i}\hbar}\langle[Q,\mathcal{H}]\rangle \qquad \frac{\mathrm{d}}{\mathrm{d}t}\langle P\rangle(t) = \frac{1}{\mathrm{i}\hbar}\langle[P,\mathcal{H}]\rangle. \tag{47}$$

By virtue of proposition 1, (45) holds. □

Following this theorem, the quantum analogue of Newton second law for expectation values leads to a nonlinear force $\mathbf{F} \propto (P^r Q + QP^r)$ which remains to be understood [26]. It turns out that this nonlinearity for a parameter r greatly affects the dynamics and generates constraints on the integrability of the wavefunction in the carrier Hilbert space.

In the following, we restrict our analysis to the case of the momentum representation. The Hamiltonian then reads

$$\mathcal{H} = \frac{\hbar\omega}{2}\left\{-\varrho_r^2\left[p^{2r}\frac{\mathrm{d}^2}{\mathrm{d}p^2} + 2rp^{2r-1}\frac{\mathrm{d}}{\mathrm{d}p} + \frac{r}{4}(3r-2)p^{2r-2}\right] + \xi^2p^2\right\}, \tag{48}$$

where $\varrho_r = \nu_r(m\omega/\hbar)^{1/2}$ and $\xi = (1/m\hbar\omega)^{1/2}$.

We note that the domain of the definition for the Sturm–Liouville operator (48) is defined as [24]

$$\begin{aligned}\mathfrak{D}(\mathcal{H}) = \{&\phi \in L^2(]0,+\infty[);\ \phi,\ p^{2r}\phi' \in AC_{loc}(]0,+\infty[\backslash\{\varrho_r/\xi\}),\\ &\left\{-\varrho_r^2\left[p^{2r}\frac{\mathrm{d}^2}{\mathrm{d}p^2} + 2rp^{2r-1}\frac{\mathrm{d}}{\mathrm{d}p} + \frac{r}{4}(3r-2)p^{2r-2}\right] + \xi^2p^2\right\}\phi \in L^2(]0,+\infty[\mathrm{d}p)\Big\},\end{aligned} \tag{49}$$

where $AC_{\mathrm{loc}}(\Lambda)$ denotes the set of all absolutely continuous functions on Λ, Λ being an open set.

The case $r = 1$ leads to

$$\mathcal{H} = \frac{\hbar\omega}{2}\left(-\varrho_1^2 p^2 \frac{\mathrm{d}^2}{\mathrm{d}p^2} - 2\varrho_1^2 p \frac{\mathrm{d}}{\mathrm{d}p} - \frac{\varrho_1^2}{4} + \xi^2 p^2\right), \tag{50}$$

where $\varrho_1 = \nu_1(m\omega/\hbar)^{1/2}$ and $\xi = (m\hbar\omega)^{-1/2}$. We consider the appropriate space of square integrable functions $\Xi \equiv L^2(\Omega, \mathrm{d}p)$, where $\Omega =]0, +\infty[$. Since we are interested in eigenfunctions which are physically acceptable, we require these functions to be, at least, square integrable and continuous.

The reduced Hamiltonian

$$\mathcal{H}_1 := \frac{\mathcal{H}}{\hbar\omega} = \frac{1}{2}\left(-\varrho_1^2 p^2 \frac{\mathrm{d}^2}{\mathrm{d}p^2} - 2\varrho_1^2 p \frac{\mathrm{d}}{\mathrm{d}p} - \frac{\varrho_1^2}{4} + \xi^2 p^2\right) \tag{51}$$

leads to the eigenvalue problem in the usual form $\mathcal{H}_1\phi(p) = \epsilon\phi(p)$, namely,

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}p^2} + \frac{2}{p}\frac{\mathrm{d}}{\mathrm{d}p} + \left[\frac{1}{4p^2} + \frac{2\epsilon}{\varrho_1^2 p^2} + \left(\frac{\mathrm{i}\xi}{\varrho_1}\right)^2\right]\right\}\phi(p) = 0. \tag{52}$$

A general solution of (52) is given by [27]

$$\phi(p) = \frac{1}{\sqrt{p}}\mathcal{Z}_\mu\left(\frac{\mathrm{i}\xi}{\varrho_1}p\right), \tag{53}$$

where $\mu^2 = -2\epsilon\big/\varrho_1^2$, $\mathcal{Z}_\mu$ being an arbitrary cylinder function. Taking into account the singularities at the origin ($p = 0$) and relevant physical boundary conditions, two cases have to be considered.

(i) $\epsilon \geqslant 0$. We get the solution

$$\phi_1(p) = \begin{cases} \frac{1}{\sqrt{p}} H_\mu^{(1)}(\mathrm{i}) J_{-\mu}\left(\frac{\mathrm{i}\xi}{\varrho_1}p\right) & \text{if} \quad p \in \left]0, \frac{\varrho_1}{\xi}\right] \\ \frac{1}{\sqrt{p}} J_{-\mu}(\mathrm{i}) \mathrm{H}_\mu^{(1)}\left(\frac{\mathrm{i}\xi}{\varrho_1}\mathrm{p}\right) & \text{if} \quad p \in \left[\frac{\varrho_1}{\xi}, +\infty\right[\end{cases} \tag{54}$$

with $\mu = \mathrm{i}\,(2\epsilon)^{1/2}/\varrho_1$. However, this function fails to be square integrable. Indeed, the behaviour for small values of p is described by the asymptotic formula

$$\frac{1}{\sqrt{p}} J_{-\mu}\left(\frac{\mathrm{i}\xi}{\varrho_1}p\right) \simeq \frac{1}{\sqrt{p}}\frac{\left(\frac{\mathrm{i}\xi}{2\varrho_1}p\right)^{-\mu}}{\Gamma(1-\mu)}, \tag{55}$$

which is not square integrable in the neighbourhood of the origin. Therefore, (54) is not a suitable solution.

(ii) $\epsilon < 0$. The eigenfunction is then defined by

$$\phi_2(p) = \begin{cases} \frac{1}{\sqrt{p}} H_\mu^{(1)}(\mathrm{i}) J_\mu\left(\frac{\mathrm{i}\xi}{\varrho_1}p\right) & \text{if} \quad p \in \left]0, \frac{\varrho_1}{\xi}\right] \\ \frac{1}{\sqrt{p}} J_\mu(\mathrm{i}) \mathrm{H}_\mu^{(1)}\left(\frac{\mathrm{i}\xi}{\varrho_1}\mathrm{p}\right) & \text{if} \quad p \in \left[\frac{\varrho_1}{\xi}, +\infty\right[, \end{cases} \tag{56}$$

where $\mu = (2|\epsilon|)^{1/2}/\varrho_1$. Using the formulae [28]

$$J_\mu\left(\frac{\mathrm{i}\xi}{\varrho_1}p\right) = \exp\left(\frac{\mathrm{i}\mu\pi}{2}\right) I_\mu\left(\frac{\xi}{\varrho_1}p\right) \tag{57}$$

and

$$H_\mu^{(1)}\left(\frac{\mathrm{i}\xi}{\varrho_1}p\right) = \frac{2}{\mathrm{i}\pi}\exp\left(-\frac{\mathrm{i}\mu\pi}{2}\right) K_\mu\left(\frac{\xi}{\varrho_1}p\right), \tag{58}$$

where I_μ and K_μ are the modified Bessel function of the first kind and Macdonald's function, respectively. Examining the asymptotic behaviour

$$I_\mu(p) \simeq \frac{p^\mu}{2^\mu \Gamma(\mu+1)} \tag{59}$$

for small values of p,

$$K_\mu(p) \simeq \sqrt{\frac{\pi}{2p}} \exp(-p) \tag{60}$$

for large values of p, one can readily check that ϕ_2 is square integrable in Ω. Therefore, bound states are obtained and the eigenfunctions are given by (56).

Let us now consider the case $r = 2$. The Hamiltonian takes the form

$$\mathcal{H}_2 = \frac{1}{2}\left[-\varrho_2^2\left(p^4\frac{\mathrm{d}^2}{\mathrm{d}p^2} + 4p^3\frac{\mathrm{d}}{\mathrm{d}p} + 2p^2\right) + \xi^2 p^2\right]. \tag{61}$$

Following step by step the above analysis for $r = 1$, we get the $r = 2$ bound states as follows:

$$\phi(p) = \begin{cases} p^{-3/2} H_{\mu'}^{(1)}\left(\dfrac{\mathrm{i}\xi\sqrt{-2\epsilon}}{\varrho_2 p}\right) J_{\mu'}\left(\dfrac{\mathrm{i}\xi^2\sqrt{-2\epsilon}}{\varrho_2^2}\right) & \text{if} \quad p \in \left]0, \dfrac{\varrho_2}{\xi}\right] \\ p^{-3/2} J_{\mu'}\left(\dfrac{\mathrm{i}\xi\sqrt{-2\epsilon}}{\varrho_2 p}\right) H_{\mu'}^{(1)}\left(\dfrac{\mathrm{i}\xi^2\sqrt{-2\epsilon}}{\varrho_2^2}\right) & \text{if} \quad p \in \left[\dfrac{\varrho_2}{\xi}, +\infty\right[, \end{cases} \tag{62}$$

where $\mu' = \left[\left(4\xi^2 + \varrho_2^2\right)/4\varrho_2^2\right]^{1/2}$.

Larger r-values could be considered, but they no longer lead to differential equations giving solutions in terms of known special functions. Hence, in so far as we are dealing with exact analytical solutions, we do restrict our analysis to the cases $r \leqslant 2$.

4. Generalization to *n* dimensions

We now turn to extending the above formalism to n spatial dimensions in the momentum representation.

A natural generalization of algebra (13) is given by

$$[Q_i, P_j] = \mathrm{i}\nu_r \delta_{ij} \|\vec{P}\|^r \qquad i, j = 1, 2, \ldots, n, \tag{63}$$

where

$$\|\vec{P}\|^2 = \sum_{i=1}^{n} P_i^2.$$

Such a generalization supposes that n is odd and the operators Q_i and P_j act on functions defined in a domain $\Omega = (\mathbb{R}^+\backslash\{0\})^n$. The requirement

$$[P_i, P_j] = 0 \tag{64}$$

allows us to straightforwardly generalize the above-provided momentum representation to n dimensions as follows:

$$\begin{aligned} P_i\psi(p) &= p_i\psi(p) \\ Q_i\psi(p) &= \mathrm{i}\nu_r\left(\|\vec{P}\|^r\partial_{p_i} + \frac{r}{2}\|\vec{P}\|^{r-2}p_i\right)\psi(p). \end{aligned} \tag{65}$$

This fixes the commutation relations among the position operators. Explicitly, we have

$$[Q_i, Q_j] = \mathrm{i}r\nu_r\|\vec{P}\|^{r-2}(P_iQ_j - P_jQ_i) \tag{66}$$

logically leading to a 'noncommutative geometric' generalization of the position space [3].

The operators Q_i and P_i are symmetric on the domain $\mathcal{C}_0^\infty(\Omega)$ with respect to the scalar product

$$\langle\psi,\phi\rangle = \int_\Omega \psi^*(p)\phi(p)\,\mathrm{d}^n p. \tag{67}$$

The functional analysis of the operators defined in (65) is analogous to the one-dimensional situation, and can be therefore discussed following step by step the approach used in section 2.

It is worth noting that the commutation relations (63), (64), (66) do not break the rotational symmetry. In fact, the generators of rotations for the n-dimensional spaces can be expressed in terms of the position and momentum as

$$L_{ij} = \frac{\hbar}{\nu_r\|\vec{P}\|^r}(Q_iP_j - Q_jP_i) \tag{68}$$

which, in three dimensions, reduce to

$$L_k := \frac{\hbar}{\nu_r\|\vec{P}\|^r}\epsilon_{ijk}Q_iP_j, \tag{69}$$

generalizing the usual definition of the orbital angular momentum. Note that $1/\|\vec{P}\|^r$ is an unproblematic bounded operator acting on the momentum space wavefunction as multiplication $1/\|\vec{p}\|^r$. The generators of the rotations act on momentum wavefunctions as

$$L_{ij}\psi(p) = -\mathrm{i}\hbar(P_i\partial_j - P_j\partial_i)\psi(p), \tag{70}$$

where

$$[P_i, L_{jk}] = \mathrm{i}\hbar(\delta_{ki}P_j - \delta_{ij}P_k) \tag{71}$$

$$[L_{ij}, L_{kl}] = \mathrm{i}\hbar(\delta_{jk}L_{li} + \delta_{jl}L_{ik} - \delta_{ik}L_{lj} - \delta_{il}L_{jk}) \tag{72}$$

$$[Q_i, L_{jk}] = \mathrm{i}\hbar(\delta_{ik}Q_j - \delta_{ij}Q_k). \tag{73}$$

Results (70), (71), (72) and (73) exactly coincide with those given in [3] with a different deformed Heisenberg algebra.

Using (66) and (70), we obtain

$$[Q_i, Q_j] = -\frac{\mathrm{i}r\nu_r^2}{\hbar}\|\vec{P}\|^{2r-2}L_{ij} \tag{74}$$

characteristic of a noncommutative geometry. Besides, the following uncertainty relations hold:

$$\Delta Q_i\Delta P_j \geqslant \frac{\nu_r}{2}|\langle\|\vec{P}\|^r\rangle|\delta_{ij} \tag{75}$$

$$\Delta Q_i\Delta Q_j \geqslant \frac{r\nu_r^2}{2\hbar}|\langle\|\vec{P}\|^{2r-2}L_{ij}\rangle|. \tag{76}$$

In three dimensions, the previous commutation relations can be simplified into the form

$$[P_i, L_j] = \mathrm{i}\hbar\epsilon_{ijk}P_k \tag{77}$$

$$[Q_i, L_j] = \mathrm{i}\hbar\epsilon_{ijk}Q_k \tag{78}$$

$$[L_i, L_j] = \mathrm{i}\hbar\epsilon_{ijk}L_k, \tag{79}$$

where

$$L_k\psi(p) = -\mathrm{i}\hbar\epsilon_{ijk}P_i\partial_{p_j}\psi(p) \tag{80}$$

refers to the action of the angular momentum operator on wavefunctions.

5. Concluding remarks

This work was mainly focused on the generalized HA generated by a deformation of the Poisson bracket. The obtained results well yield those of usual quantum mechanics for the deformation parameter $r = 0$. The representation theory can be handled in both the configuration and momentum spaces. As an illustration, for the deforming parameter $r \leqslant 2$, we succeeded in getting well-behaved solutions in the momentum space for the harmonic oscillator. The Hamiltonian eigenfunctions are expressed in terms of cylinder functions.

As a matter of qualitative analysis, the generalized commutation relations (13) can be rewritten for unitless operators Q and P in the form

$$[Q, P] = \mathrm{i}P^r \tag{81}$$

and the Hamiltonian of the harmonic oscillator in this case reduces to

$$H = \tfrac{1}{2}(P^2 + Q^2). \tag{82}$$

Evidently, in the case $r = 1$, in the momentum representation for instance, the operators

$$Q = \mathrm{i}\left(p\frac{\mathrm{d}}{\mathrm{d}p} + \frac{1}{2}\right) \qquad P = p \tag{83}$$

are unitary equivalent to

$$\tilde{Q} = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}p} \qquad \tilde{P} = \exp(p) \tag{84}$$

with the unitary operator explicitly given by

$$\begin{aligned} &U : L^2(\mathbb{R}^2) \longrightarrow L^2(\mathbb{R}^2, \mathbb{C}^2) \\ &U\phi(p) = \exp(p/2)(\phi(\exp(p)); \phi(-\exp(p))) \qquad \forall\phi \in L^2(\mathbb{R}^2). \end{aligned} \tag{85}$$

By analogy, the Hamiltonian (82) is unitary equivalent to the Hamiltonian

$$\tilde{H} = \frac{1}{2}\left(-\frac{\mathrm{d}^2}{\mathrm{d}p^2} + \exp(2p)\right). \tag{86}$$

Finally, let us mention that the eigenvalue problem for the oscillator (44) in the configuration space reveals cumbersome measure problem which occurs when one deals with the square integrability of the eigenfunctions. A thorough analysis of these aspects, including the investigation of related relevant properties as well as the physical regime in which the deformed relations could be approximately the canonical relations, is in progress and will be at the core of a forthcoming paper.

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